

LIPSCHITZ STABILITY OF AN INVERSE BOUNDARY VALUE PROBLEM FOR A SCHRÖDINGER TYPE EQUATION

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Abstract. In this paper we study the inverse boundary value problem of determining the potential in the Schrödinger equation from the knowledge of the Dirichlet-to-Neumann map, which is commonly accepted as an ill-posed problem in the sense that, under general settings, the optimal stability estimate is of logarithmic type. In this work, a Lipschitz type stability is established assuming a priori that the potential is piecewise constant with a bounded known number of unknown values.

1. Introduction. In this paper, we investigate the stability for the inverse boundary value problem of a Schrödinger equation with complex potential, $q(x)$ say. This encompasses the Helmholtz equation with attenuation, when $q(x) = \omega^2 c^{-2}(x)$, where c denotes the speed of propagation and ω is the frequency, which can be complex. In fact, the imaginary part of $\omega c^{-1}(x)$ characterizes the attenuation of waves in the medium.

We begin with formulating the direct problem. Let $u \in H^1(\Omega)$ be the weak solution to the boundary value problem,

$$(1.1) \quad \begin{cases} (-\Delta + q(x))u = 0, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded connected domain, $q \in L^\infty(\Omega)$ is a complex-valued function and g is prescribed in the trace space $H^{1/2}(\Omega)$. The Dirichlet-to-Neumann map is the operator $\Lambda_q : H^{1/2}(\Omega) \rightarrow H^{-1/2}(\Omega)$ given by

$$(1.2) \quad g \rightarrow \Lambda_q g = \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega},$$

where ν is the exterior unit normal vector to $\partial\Omega$.

The inverse problem that we consider, consists in determining q when Λ_q is known. This problem arises in geophysics, for example, in reflection seismology assuming a description in terms of time-harmonic scalar waves. The topic of this paper is the issue of continuous dependence of q from the Dirichlet-to-Neumann map Λ_q . The continuous dependence is of fundamental importance for the robustness of any reconstruction, as well as for the development of convergent iterative reconstruction procedures starting not too far from the solution (cf. [5]). More precisely, it has been proved that Landweber iteration reconstruction methods converge if the continuous dependence for the inverse problem is of Hölder or Lipschitz type.

From the work of [10], it is evident that for arbitrary potentials q , Lipschitz stability cannot hold. Motivated by, and following analogous results in electrical impedance tomography (EIT, cf. [3, 4]), here we study conditional stability when a-priori information on q is assumed. We consider models with discontinuous potentials to accommodate realistic reflectors. Specifically, we consider the space spanned by linear

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combinations of N characteristic functions. More precisely we consider potentials of the form

$$q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x),$$

where $q_j, j = 1, \dots, N$ are unknown complex numbers and D_j are known open Lipschitz sets in \mathbb{R}^n . Moreover, we consider the case of partial boundary data, that is, we can restrict the collection of measurements to only a part of the boundary. We refer to [13] for a review of recent uniqueness results. Here, we prove Lipschitz stability with a uniform constant, which depends on N and on the other a-priori parameters of the problem. We will show that the Lipschitz constant grows exponentially with the dimension, N , of the space of potentials. The method of proof follows the ideas introduced in Alessandrini and Vessella and relies on quantitative estimates of unique continuation of solutions to elliptic systems and on the use of singular solutions and of their asymptotic behaviour near the discontinuity interfaces. Compared to the case of the real or complex conductivity equation in the case of the Schrödinger equation we are able to derive our result relaxing the assumptions of regularity on ∂D_j that are assumed to be Lipschitz. Furthermore, taking advantage of the regularity of solutions and of its gradient inside the domain Ω we find a better dependence of the stability constant on N .

The outline of the paper is as follows. In the next section we state all the assumptions and the main result. In Section 3, we give a summary of known regularity results connected to Schrödinger equation with complex potential, and some preparatory lemmas concerning the existence and asymptotic behaviour of singular solutions. Section 4 contains the proof of our main theorem. We first show the proof for $n = 3$ and then modify it to the other cases. For the structure of the main proof we characterize the rate of blow-up of the singular functions finding lower and upper bounds in terms of the distance of the singularity from the interface of the subdomains. More precisely, to derive our main result we first establish that the singular function satisfies a lower bound in terms of the distance of the singularity from the interface. Secondly, by using quantitative estimates of propagation of smallness we derive also an upper bound for the singular function. Last but not least, we make use the value of a bounded non-decreasing function at some particular point to prove that either the result of the main theorem can be deduced directly or a recursive inequality (4.22) must hold true. The recursive inequality also leads to the desired result. In Section 5 we demonstrate by an example that the Lipschitz constant grows exponentially with the dimension of the space of potentials. This example is constructed from its analogue in electrical impedance tomography [11].

2. Main result.

2.1. Notation and definitions. We denote by n the space dimension. For every $x \in \mathbb{R}^n$, we set $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ for $n \geq 2$. With $B_R(x)$, $B'_R(x')$ and $Q_R(x)$ we denote the open ball in \mathbb{R}^n centered at x of radius R , the ball in \mathbb{R}^{n-1} centered at x' of radius R , and the cylinder $B'_R(x') \times (x_n - R, x_n + R)$, respectively. For simplicity of notation, $B_R(0)$, $B'_R(0)$ and $Q_R(0)$ are denoted by B_R , B'_R and Q_R .

DEFINITION 2.1. *Let Ω be a bounded domain in \mathbb{R}^n . We say that a portion Σ of $\partial\Omega$ is of Lipschitz class with constants $r_0, L > 0$ if, for any $P \in \Sigma$, there exists a*

rigid transformation of coordinates such that $P = 0$ and

$$\Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} \mid x_n > \phi(x')\}$$

where ϕ is a Lipschitz continuous function on B'_{r_0} with $\phi(0) = 0$ and

$$\|\phi\|_{C^{0,1}(B'_{r_0})} \leq L.$$

We shall say that Ω is of Lipschitz class with constants r_0 and L , if $\partial\Omega$ is of Lipschitz class with the same constants.

DEFINITION 2.2. Let Ω be a bounded open subset of \mathbb{R}^n and of Lipschitz class and Σ be a open portion of $\partial\Omega$. We define $H_{co}^{1/2}(\Sigma)$ as

$$H_{co}^{1/2}(\Sigma) = \{g \in H^{1/2}(\partial\Omega) \mid \text{supp } g \subset \Sigma\}$$

and $H_{co}^{-1/2}(\Sigma)$ as the topological dual of $H_{co}^{1/2}(\Sigma)$; we denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $H_{co}^{1/2}(\Sigma)$ and $H_{co}^{-1/2}(\Sigma)$.

DEFINITION 2.3. Let Ω be a bounded open subset of \mathbb{R}^n and of Lipschitz class, Σ be a open portion of $\partial\Omega$ and $q \in L^\infty(\Omega)$. Assume that 0 is not an eigenvalue of $(-\Delta + q)$ with Dirichlet boundary conditions in Ω , i.e.,

$$\{u \in H_0^1(\Omega) \mid (-\Delta + q)u = 0\} = \{0\}.$$

For any $g \in H_{co}^{1/2}(\Sigma)$, let $u \in H^1(\Omega)$ be the weak solution to the Dirichlet problem

$$(2.1) \quad \begin{cases} (-\Delta + q(x))u = 0, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases}$$

We define the local Dirichlet-to-Neumann map $\Lambda_q^{(\Sigma)}$ as

$$\Lambda_q^{(\Sigma)} : H_{co}^{1/2}(\Sigma) \rightarrow H_{co}^{-1/2}(\Sigma) \\ g \mapsto \left. \frac{\partial u}{\partial \nu} \right|_{\Sigma},$$

where ν is the exterior unit normal vector to $\partial\Omega$.

With Ω being a bounded open set, with $C^{0,1}$ boundary, the set of the eigenvalues of $(-\Delta + q)$ with Dirichlet boundary conditions is a discrete subset of \mathbb{C} , and hence can be avoided.

We observe that $\Lambda_q^{(\Sigma)}$ can be identified with the sesquilinear form on $H_{co}^{1/2}(\Sigma) \times H_{co}^{1/2}(\Sigma)$, defined by

$$\langle \Lambda_q^{(\Sigma)} g, f \rangle = \int_{\Omega} (\nabla u \cdot \nabla \bar{v} + qu\bar{v}) dx, \quad \forall f, g \in H_{co}^{1/2}(\Sigma),$$

where u is the solution to (2.1) and v is any function in $H^1(\Omega)$ such that $v|_{\partial\Omega} = f$. This definition is independent of the choice of v : Let v_1, v_2 be two different functions

in $H^1(\Omega)$ such that $v_1|_{\partial\Omega} = v_2|_{\partial\Omega} = f$. Then, since $w = v_1 - v_2 \in H_0^1(\Omega)$, and u is a solution, we have

$$\begin{aligned} \int_{\Omega} (\nabla u \cdot \nabla \bar{v}_1 + qu\bar{v}_1) dx - \int_{\Omega} (\nabla u \cdot \nabla \bar{v}_2 + qu\bar{v}_2) dx \\ = \int_{\Omega} (\nabla u \nabla \bar{w} + qu\bar{w}) dx = 0, \end{aligned}$$

using integration by parts. We denote by $\|\cdot\|_{\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))}$ the norm defined as

$$\|\Lambda_q^{(\Sigma)}\|_{\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))} = \sup_{f, g \in H_{co}^{1/2}(\Sigma)} \{ \langle \Lambda_q^{(\Sigma)} g, f \rangle \mid \|g\|_{H_{co}^{1/2}(\Sigma)} = \|f\|_{H_{co}^{1/2}(\Sigma)} = 1 \}.$$

2.2. Main assumptions. Our assumptions on Ω and $q(x)$ are

ASSUMPTION 2.4. $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying

$$|\Omega| \leq A$$

Here and in the sequel $|\Omega|$ denotes the Lebesgue measure of Ω . We assume that $\partial\Omega$ is of Lipschitz class and we fix an open portion Σ of $\partial\Omega$ which is of Lipschitz class with constants r_0 and L .

ASSUMPTION 2.5. The complex-valued function $q(x)$ satisfies

$$\|q\|_{L^\infty(\Omega)} \leq B,$$

where B is a positive constant, and is of the form

$$q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x),$$

where $q_j, j = 1, \dots, N$ are unknown complex numbers and D_j are known open sets in \mathbb{R}^n which satisfy the following assumption. Moreover, we assume that 0 is not an eigenvalue of $-(\Delta + q)$ with Dirichlet boundary conditions in Ω .

ASSUMPTION 2.6. The $D_j, j = 1, \dots, N$, are connected and pairwise non-overlapping open sets such that $\cup_{j=1}^N \overline{D_j} = \overline{\Omega}$ and ∂D_j are of Lipschitz class. We also assume that there exists one set, say D_1 , such that $\partial D_1 \cap \partial\Omega$ contains an open portion Σ_1 of Lipschitz class with constants r_0 and L . For every $j \in \{2, \dots, N\}$ there exist $j_1, \dots, j_M \in \{1, \dots, N\}$ such that

$$D_{j_1} = D_1, \quad D_{j_M} = D_j$$

and, for every $k = 1, \dots, M$,

$$\partial D_{j_{k-1}} \cap \partial D_{j_k}$$

contains a non-empty open portion Σ_k of Lipschitz class with constants r_0 and L such that

$$\begin{aligned} \Sigma_1 &\subset \Sigma, \\ \Sigma_k &\subset \Omega, \quad \forall k = 2, \dots, M. \end{aligned}$$

Furthermore, there exists $P_k \in \Sigma_k$, at which D_{k-1} satisfies the interior ball condition with radius $\frac{3r_0}{16}$, and a rigid transformation of coordinates such that $P_k = 0$ and

$$\begin{aligned}\Sigma_k \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n = \phi_k(x')\}, \\ D_{j_k} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n > \phi_k(x')\}, \\ D_{j_{k-1}} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} \mid x_n < \phi_k(x')\},\end{aligned}$$

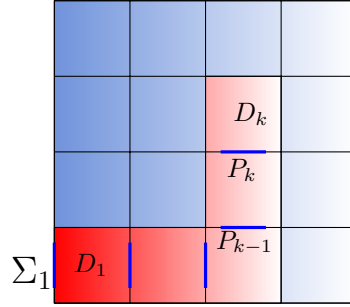
where ϕ_k is a $C^{0,1}$ function on $B'_{r_0/3}$ satisfying

$$\phi_k(0) = 0$$

and

$$\|\phi_k\|_{C^{0,1}(B'_{r_0/3})} \leq L.$$

For simplicity, we call D_{j_1}, \dots, D_{j_M} a chain of domains connecting D_1 to D_j .



In the further analysis, for simplicity of notation, we also use the constant $r_1 = \frac{r_0}{16}$.

2.3. Statement of the main result. The main result of this paper is stated as follows.

THEOREM 2.7. *Let Ω satisfy Assumption 2.4 and $q^{(k)}, k = 1, 2$ be two complex piecewise constant functions of the form*

$$q^{(k)}(x) = \sum_{j=1}^N q_j^{(k)} \chi_{D_j}(x), \quad k = 1, 2$$

which satisfy Assumption 2.5 and $D_j, j = 1, \dots, N$ satisfy Assumption 2.6. Then, there exists a constant $C = C(n, r_0, L, A, B, N)$, such that

$$(2.2) \quad \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} \leq C \|\Lambda_1^{(\Sigma)} - \Lambda_2^{(\Sigma)}\|_{\mathcal{L}(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma))},$$

where $\Lambda_k^{(\Sigma)} = \Lambda_{q^{(k)}}^{(\Sigma)}$ for $k = 1, 2$.

3. Preliminary results. In this section, we state some results which will be used in the proof of our main stability result.

PROPOSITION 3.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n , $q \in L^\infty(\Omega)$ complex valued potential, $f \in L^p(\Omega)$ and $g \in W^{2-\frac{1}{p},p}(\partial\Omega)$ with $1 < p < \infty$. Assume that 0 is not a Dirichlet eigenvalue for the operator $-\Delta + q$ in Ω . Then there exists a unique solution $u \in W^{2,p}(\Omega)$ to the problem*

$$(3.1) \quad \begin{cases} (-\Delta + q(x))u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases}$$

Moreover,

$$(3.2) \quad \|u\|_{W^{2,p}(\Omega)} \leq C \left(\|g\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|f\|_{L^p(\Omega)} \right)$$

where C depends on n, Ω and $\|q\|_{L^\infty(\Omega)}$.

The proof is a consequence of the existence of a $W^{2,p}(\Omega)$ function w such that $w = g$ on $\partial\Omega$ and such that $\|w\|_{W^{2,p}(\Omega)} \leq C\|g\|_{W^{2-\frac{1}{p},p}(\partial\Omega)}$ and of the Fredholm alternative; see for example Theorem 3.5.8 in Feldman and Uhlmann's notes [7]). For reader's convenience, we also note the following Proposition 3.2 without proof, which we use for the low dimension cases.

PROPOSITION 3.2. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n , $q \in L^\infty(\Omega)$ complex valued potential, $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. Assume that 0 is not a Dirichlet eigenvalue for the operator $-\Delta + q$ in Ω . Then there exists a unique solution $u \in H^1(\Omega)$ to the equation (3.1). Moreover,*

$$(3.3) \quad \|u\|_{H^1(\Omega)} \leq C (\|g\|_{H^{1/2}(\partial\Omega)} + \|f\|_{H^{-1}(\Omega)})$$

where C depends on n, Ω and $\|q\|_{L^\infty(\Omega)}$.

Our approach follows the one of Beretta and Francini[4], which is for the EIT problem with complex conductivity, of constructing singular solutions and of studying their asymptotic behavior when the singularity approaches the interfaces Σ_k . This method was originally introduced by Alessandrini and Vessella in the real-valued conductivity case [3]. To construct singular solutions for the EIT problems, the Green's function plays a crucial role. In our case, we also use the Green's function to treat the case of high dimension ($n \geq 4$) and a first order derivative of Green's function needs to be used for lower dimension ($n = 2, 3$). In the following propositions, we discuss the existence and behavior of the Green's functions ($n \geq 4$) and a first order derivative of the Green's function ($n = 2, 3$) when q satisfies Assumption 2.5. We are especially interested in their asymptotic behavior near the $C^{0,1}$ interface Σ_k .

Before doing this, we need to extend our original domain. We consider Σ_1 and recall that up to a rigid transformation of coordinates we can assume that $P_1 = 0$ and

$$(\mathbb{R}^n \setminus \Omega) \cap B_{r_0} = \{(x', x_n) \in B_{r_0} \mid x_n < \phi(x')\}$$

where ϕ is a Lipschitz function such that $\phi(0) = 0$ and $\|\phi\|_{C^{0,1}(B'_{r_0})} \leq L$. Then we extend Ω to $\Omega_0 = \Omega \cup D_0$ by adding an open set D_0 defined as

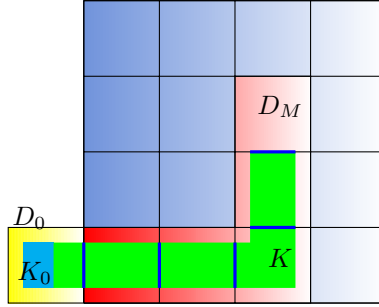
$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \mid \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6}r_0, |x_i| < \frac{2}{3}r_0, i = 1, \dots, n-1 \right\}.$$

It turns out that Ω_0 is of Lipschitz class with constants $\frac{r_0}{3}$ and L_1 , where L_1 depends on L only. We define

$$K_0 = \left\{ x \in D_0 \mid \text{dist}(x, \Sigma_1) \geq \frac{r_0}{3} \right\}$$

with $\text{dist}(K_0, \partial\Omega) > \frac{r_0}{3}$. We extend $q(x)$ defined on Ω by setting it equal to 1 in D_0 . For simplicity of notation we still denote this extension by $q(x)$.

We consider any subdomain in Ω and the chain of domains connecting it to D_1 . For simplicity let us rearrange the indices of subdomains so that this chain corresponds to D_0, D_1, \dots, D_M , $M \leq N$. Let $S = \cup_{j=0}^M \overline{D_j}$ and K be a connected subset of S with Lipschitz boundary such that $\overline{K} \cap \partial D_j = \Sigma_j \cup \Sigma_{j+1}$ for $j = 1, 2, \dots, M$, $K_0 \subset K$ and $\text{dist}(K, \partial S \setminus \{\Sigma_{M+1} \cup \Sigma_1\}) > \frac{r_0}{16}$.



In the following, we shall use C to denote positive constants. The value of the constants may change from line to line, but we shall specify their dependence everywhere where they appear. For $n \geq 4$, let Γ denote the fundamental solution associated with the Laplace operator. In the proof of Theorem 2.7, we will need to estimate $G - \Gamma$ from above in terms of variable-interface distance r to a power, which is smaller than the order of the singularity of Γ . Since, for high dimension cases ($n \geq 6$), $\Gamma(\cdot, y)$ does not belong to $H^{-1}(\Omega)$, we need to employ L^p estimate of the solutions here. Note that $\Gamma(\cdot, y)$ belongs to $L^p(\Omega)$ for any $1 \leq p < \frac{n}{n-2}$.

PROPOSITION 3.3. *Let the complex-valued function $q \in L^\infty(\Omega_0)$ satisfy Assumption 2.5 and $n \geq 4$. For $y \in \Omega_0$, there exists a unique function $G(\cdot, y)$ continuous in $\Omega_0 \setminus \{y\}$ such that*

$$(3.4) \quad \int_{\Omega_0} \nabla G(\cdot, y) \nabla \phi + q G(\cdot, y) \phi = \phi(y), \quad \forall \phi \in C_0^\infty(\Omega).$$

Furthermore, we have that $G(x, y)$ is symmetric, that is,

$$(3.5) \quad G(x, y) = G(y, x), \quad x, y \in \Omega_0,$$

and the following estimates

$$(3.6) \quad \begin{aligned} \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} &\leq C |\ln r|^{\frac{1}{2}}, & r &\leq \frac{1}{2} \text{dist}(y, \partial\Omega_0), & n &= 4 \\ \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} &\leq C r^{2-\frac{n}{2}}, & r &\leq \frac{1}{2} \text{dist}(y, \partial\Omega_0), & n &\geq 5 \end{aligned}$$

and

$$(3.7) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq \begin{cases} C & , \quad 4 \leq n \leq 7, \\ |\ln(\text{dist}(y, \cup_{j=1}^N \partial D_j))| & , \quad n = 8, \\ \text{dist}(y, \cup_{j=1}^N \partial D_j)^{4-\frac{n}{2}} & , \quad n \geq 9, \end{cases}$$

for $\text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}$, hold true, where the constant C depends on the constant in Proposition 3.1.

Proof. Assume that y belongs to some sub-domain D_m which q equals to a complex constant q_m inside. Let $H(x, y)$ denote the outgoing fundamental solution of Helmholtz equation

$$(-\Delta + q_m)H(x, y) = \delta(x, y), \quad x \in \mathbb{R}^n,$$

i.e.,

$$(3.8) \quad H(x, y) = \frac{q_m^{(n-2)/4} H_{(n-2)/2}^{(1)}(q_m^{1/2}|x-y|)}{4i(2\pi)^{(n-2)/2}|x-y|^{(n-2)/2}},$$

where $H_n^{(1)}$ denotes Hankel function of the first kind. We consider $G(x, y) = H(x, y) + \omega(x, y)$, where ω solves

$$(3.9) \quad \begin{cases} (-\Delta + q)\omega = (q_m - q)H, & \text{in } \Omega_0, \\ \omega = -H, & \text{on } \partial\Omega_0. \end{cases}$$

Note that $q_m - q$ vanishes in D_m . Hence $(q_m - q)H$ belongs to $L^\infty(\Omega_0)$. By using the asymptotic behavior of the Hankel function near the origin [12], we obtain that

$$|(q_m - q)H(x, y)| \leq \begin{cases} 0 & , \quad |x - y| \leq \text{dist}(y, \partial D_m), \\ C|x - y|^{2-n} & , \quad |x - y| > \text{dist}(y, \partial D_m), \end{cases}$$

for some positive constant C . We observe that the order of the singularity of $\omega(x, y)$ is always lower than the fundamental solution $H(x, y)$. To be more precise, by applying Proposition 3.1 with $p = \frac{2n}{n+4}$ and Sobolev embedding theorem, we conclude that

$$(3.10) \quad \begin{aligned} \|\omega(\cdot, y)\|_{L^2(\Omega_0)} &\leq C\|\omega(\cdot, y)\|_{W^{2, \frac{2n}{n+4}}(\Omega_0)} \\ &\leq C\|(q_m - q)H(\cdot, y)\|_{L^{\frac{2n}{n+4}}(\Omega_0)} \leq \begin{cases} C & , \quad 4 \leq n \leq 7, \\ |\ln(\text{dist}(y, \partial D_m))| & , \quad n = 8, \\ \text{dist}(y, \partial D_m)^{4-\frac{n}{2}} & , \quad n \geq 9. \end{cases} \end{aligned}$$

Then, using the asymptotic behavior of the Hankel function again and the inequality

$$\|G\|_{L^2(\Omega_0 \setminus B_r(y))} \leq \|\omega\|_{L^2(\Omega_0 \setminus B_r(y))} + \|H\|_{L^2(\Omega_0 \setminus B_r(y))}$$

we immediately get (3.6).

Let $\tilde{\Gamma}(\cdot)$ stand for the Gamma function. Noting that

$$H(\cdot, y), \Gamma(\cdot, y) \in C^\infty(\Omega_0 - \{y\})$$

and

$$\begin{aligned} & H(x, y) - \Gamma(x, y) \\ & \sim -\frac{i}{\pi} \tilde{\Gamma} \left(\frac{n-2}{2} \right) \frac{1}{4i \pi^{(n-2)/2}} |x-y|^{2-n} - \frac{\tilde{\Gamma} \left(\frac{n+2}{2} \right)}{n(2-n) \pi^{n/2}} |x-y|^{2-n} \\ & = 0, \end{aligned}$$

as $|x-y|$ goes to 0, we conclude that $|\Gamma(\cdot, y) - H(\cdot, y)|$ is uniformly bounded for all y such that $\text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}$. Then (3.7) follows. \square

In both Beretta & Francini's proof [4] and Alessandrini & Vessella's proof [3], the blow-up property of a singular function,

$$\int_{U_k} \nabla G_1(y, x) \nabla G_2(x, y) dx,$$

where $U_k = \Omega \setminus \cup_{j=1}^k D_j$ and G_1, G_2 are functions defined by (3.4) for potentials $q^{(1)}, q^{(2)}$, respectively, when y approaches the interfaces, is essential. However, in the case of the Schrödinger equation, this does not happen if $n = 2, 3$. Therefore, for $n = 2, 3$, we will introduce a derivative in the point source. For $n = 3$, let

$$\Gamma(x, y) = -\frac{x_3 - y_3}{4\pi|x-y|^3},$$

which is the solution to the equation

$$(3.11) \quad -\Delta \Gamma(x, y) = \frac{\partial}{\partial x_3} \delta_y(x).$$

PROPOSITION 3.4. *Let $n = 3$ and $q \in L^\infty(\Omega_0)$. For $y \in \Omega_0$, there exists a unique function $G(\cdot, y)$ continuous in $\Omega_0 \setminus \{y\}$ such that*

$$(3.12) \quad \int_{\Omega_0} \nabla G(\cdot, y) \cdot \nabla \phi + qG(\cdot, y)\phi = \frac{\partial}{\partial x_n} \phi(y), \quad \forall \phi \in C_0^\infty(\Omega).$$

Furthermore, we have that $G(x, y)$ is symmetric, i.e.,

$$(3.13) \quad G(x, y) = G(y, x), \quad x, y \in \Omega_0,$$

and the following estimates

$$(3.14) \quad \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq Cr^{-\frac{1}{2}}, \quad r \leq \frac{1}{2} \text{dist}(y, \partial\Omega)$$

and

$$(3.15) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq C, \quad \text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}.$$

hold, where the constant C depends on the constant in Proposition 3.2.

Proof. Consider $G(x, y) = \Gamma(x, y) + \omega(x, y)$, where ω solves

$$(3.16) \quad \begin{cases} (-\Delta + q)\omega = & q\Gamma, & \text{in } \Omega_0, \\ \omega = & -\Gamma, & \text{on } \partial\Omega_0. \end{cases}$$

Since $\Gamma(\cdot, y) \in W^{5/4, 4/3}(\partial\Omega_0)$, $q\Gamma \in L^{4/3}(\Omega_0)$ and $-\Gamma(\cdot, y) \in H^{1/2}(\partial\Omega_0)$, by Proposition 3.2, (3.16) has a unique solution $\omega \in H^1(\Omega_0)$ and $\omega = G - \Gamma$ satisfies the estimate

$$(3.17) \quad \|\omega(\cdot, y)\|_{H^1(\Omega_0)} \leq C \left(\|\Gamma(\cdot, y)\|_{H^{1/2}(\partial\Omega_0)} + \|q(\cdot)\Gamma(\cdot, y)\|_{H^{-1}(\Omega_0)} \right) \leq C,$$

when $\text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}$. Hence,

$$(3.18) \quad \begin{aligned} \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} &= \|\omega(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \\ &\leq \|\omega(\cdot, y)\|_{H^1(\Omega_0 \setminus B_r(y))} \leq \|\omega(\cdot, y)\|_{H^1(\Omega_0)} \leq C. \end{aligned}$$

With the fact that

$$(3.19) \quad \|\Gamma(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq Cr^{-\frac{1}{2}}, \quad r \leq \frac{1}{2} \text{dist}(y, \partial\Omega),$$

(3.18) gives the desired estimate

$$(3.20) \quad \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq Cr^{-\frac{1}{2}}, \quad r \leq \frac{1}{2} \text{dist}(y, \partial\Omega).$$

Finally, again by (3.17) we immediately get

$$(3.21) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq C.$$

□

For $n = 2$, let

$$\Gamma(x, y) = -\frac{2\pi(x_2 - y_2)}{|x - y|^2}$$

which is the solution to the equation

$$(3.22) \quad -\Delta\Gamma(x, y) = \frac{\partial}{\partial x_2} \delta_y(x).$$

PROPOSITION 3.5. *Let $n = 2$ and $q \in L^\infty(\Omega_0)$. For $y \in \Omega_0$, there exists a unique function $G(\cdot, y)$ continuous in $\Omega_0 \setminus \{y\}$ such that*

$$(3.23) \quad \int_{\Omega_0} (\nabla G(\cdot, y) \cdot \nabla \phi + qG(\cdot, y)\phi) = \frac{\partial}{\partial x_n} \phi(y), \quad \forall \phi \in C_0^\infty(\Omega).$$

Furthermore, we have that $G(x, y)$ is symmetric, that is,

$$(3.24) \quad G(x, y) = G(y, x), \quad x, y \in \Omega_0,$$

and the estimates

$$(3.25) \quad \|G(\cdot, y)\|_{L^2(\Omega_0 \setminus B_r(y))} \leq C |\ln r|^{\frac{1}{2}}, \quad r \leq \frac{1}{2} \min \left(\text{dist}(y, \partial\Omega_0), \frac{1}{2} \right)$$

and

$$(3.26) \quad \|G(\cdot, y) - \Gamma(\cdot, y)\|_{L^2(\Omega_0)} \leq C, \quad \text{dist}(y, \partial\Omega_0) \geq \frac{r_0}{16}$$

hold, where the constant C depends on the constant in Proposition 3.2.

We omit the proof here, because it follows from an adaption of the proof of Proposition 3.4. The symmetry of G follows by standard arguments based on integration by parts (see for example [6]).

In the sequel we will derive estimates of unique continuation in K for solutions to our equation. A key ingredient to obtain these estimates is the Three Spheres Inequality that we will state below and that was proved by [1, Theorem 3.1]. The next two propositions concern Three Sphere Inequalities for our equation. To prove it, one interprets the equation $(-\Delta + q)u = 0$ for a complex function $q(x)$ as a weakly coupled system of equations with Laplacian principal part

$$(3.27) \quad -\Delta U + QU = 0,$$

where U is a vector with components the real and imaginary parts of u , that is, $u^{(1)} = \Re u$, $u^{(2)} = \Im u$, and Q is a two by two tensor with elements the real and complex part of the potential q , that is, $q^{(1)} = \Re q$ and $q^{(2)} = \Im q$. We can also write the system in the form

$$\begin{cases} -\Delta u^{(1)} + q^{(1)}u^{(1)} - q^{(2)}u^{(2)} &= 0, \\ -\Delta u^{(2)} + q^{(1)}u^{(2)} + q^{(2)}u^{(1)} &= 0. \end{cases}$$

In [1, Theorem 3.1] the authors prove the validity of the Three spheres inequality for elliptic systems with Laplacian principal part. In particular it applies to solutions U of (3.27) and hence also to solutions of $(-\Delta + q)u = 0$.

PROPOSITION 3.6. *Let u be a solution to the equation*

$$(-\Delta + q)u = 0 \quad \text{in } B_R.$$

Then, for every ρ_1, ρ_2, ρ_3 , with $0 < \rho_1 < \rho_2 < \rho_3 \leq R$,

$$(3.28) \quad \|u\|_{L^2(B_{\rho_2})} \leq Q_2 \|u\|_{L^2(B_{\rho_1})}^\alpha \|u\|_{L^2(B_{\rho_3})}^{1-\alpha},$$

where $\alpha = \frac{\ln \frac{\rho_2}{\rho_3}}{\ln \frac{\rho_2}{\rho_1}} \in (0, 1)$ and $Q_2 \geq 1$ depends on $\|q\|_{L^\infty(B_R)}$, $\frac{\rho_2}{\rho_1}$ and $\frac{\rho_3}{\rho_2}$.

REMARK 3.7. *In [1, Theorem 3.1] the authors prove the validity of the three-spheres inequality for elliptic systems with some limitations on the radii. The derivation of the inequality for arbitrary radii follows by applying the argument of the proof of [2, Theorem 5.1] choosing $B_{r_0}(x_0) = B_{r_1}$, $G = B_{r_2}$ and $\Omega = B_{r_3}$.*

Also, we have

COROLLARY 3.8. *Let u be a solution to the equation*

$$(-\Delta + q)u = 0 \quad \text{in } B_R.$$

Then, for every ρ_1, ρ_2, ρ_3 , with $0 < \rho_1 < \rho_2 < \rho_3 \leq R$,

$$(3.29) \quad \|u\|_{L^\infty(B_{\rho_2})} \leq Q_\infty \|u\|_{L^\infty(B_{\rho_1})}^\beta \|u\|_{L^\infty(B_{\rho_3})}^{1-\beta},$$

where $\beta = \frac{\ln \frac{2\rho_3}{\rho_2+\rho_3}}{\ln \frac{\rho_3}{\rho_1}} \in (0, 1)$ and $Q_\infty \geq 1$ depends on $\|q\|_{L^\infty(B_R)}$, $\frac{\rho_2}{\rho_1}$ and $\frac{\rho_3}{\rho_2}$.

Proof. We use the local boundedness estimate for $u^{(1)}$ and $u^{(2)}$, weak solutions of elliptic equations (see for instance [8, Theorem 8.17]), to obtain that there exists a constant C , which only depends on n and $\|q\|_{L^\infty(B_R)}$, such that

$$(3.30) \quad \|u\|_{L^\infty(B_{\rho_2})} \leq \frac{C}{(\rho_3 - \rho_2)^{n/2}} \|u\|_{L^2(B_{\rho_3})}.$$

Then, by Proposition 3.6,

$$(3.31) \quad \begin{aligned} \|u\|_{L^\infty(B_{\rho_2})} &\leq \frac{C}{\left(\frac{\rho_2+\rho_3}{2} - \rho_2\right)^{n/2}} \|u\|_{L^2(B_{\frac{\rho_2+\rho_3}{2}})} \\ &\leq \frac{CQ_2}{\left(\frac{\rho_2+\rho_3}{2} - \rho_2\right)^{n/2}} \|u\|_{L^2(B_{\rho_1})}^\alpha \|u\|_{L^2(B_{\rho_3})}^{1-\alpha} \\ &\leq \frac{CQ_2}{\left(\frac{\rho_2+\rho_3}{2} - \rho_2\right)^{n/2}} |B_{\rho_1}|^{\alpha/2} |B_{\rho_3}|^{(1-\alpha)/2} \|u\|_{L^\infty(B_{\rho_1})}^\alpha \|u\|_{L^\infty(B_{\rho_3})}^{1-\alpha}. \end{aligned}$$

□

As a consequence of the Three Spheres Inequality stated in Corollary 3.8, we derive the following quantitative estimate for unique continuation of solutions to our equation.

PROPOSITION 3.9. *Let K and K_0 be defined as before, and let $v \in H^1(K)$ be a weak solution to the equation*

$$(-\Delta + q(x))v = 0 \quad \text{in } K.$$

Assume that, for given positive numbers ε_0 , E_0 and real number γ , v satisfies

$$(3.32) \quad \|v\|_{L^\infty(K_0)} \leq \varepsilon_0,$$

and

$$(3.33) \quad |v(x)| \leq (\varepsilon_0 + E_0) \operatorname{dist}(x, \Sigma_{M+1})^\gamma, \quad x \in K.$$

Then the following inequality holds true for every $0 < r < 2r_1$,

$$(3.34) \quad |v(\tilde{x})| \leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E_0) r^{(1-\tau_r)\gamma},$$

where $\tilde{x} = P_{M+1} - r\nu(P_{M+1})$ with ν being the exterior unit normal vector to ∂D_M at P_{M+1} , $\beta = \frac{\ln(8/7)}{\ln 4}$, $\tau_r = \frac{\ln \left(\frac{12r_1 - 2r}{12r_1 - 3r} \right)}{\ln \left(\frac{6r_1 - r}{2r_1} \right)} \in (0, 1)$ and the constants N_1 and C depend on r_0, L, A, B and n .

Proof. We construct a chain of spheres of radius r_1 with centers x_0, x_1, \dots, x_k such that the first is $B_{r_1}(x_0) \subset B_{4r_1}(x_0) \subset K_0$, all the spheres are externally tangent, and the last one is centered at $x_k = P_{M+1} - 3r_1\nu(P_{M+1})$. We choose this chain so

that the spheres of radius $4r_1$ concentric with those of the chain, except the last one, are contained in K and have a distance greater than r_1 away from Σ_{M+1} . Such a chain has a finite number of spheres that is smaller than $N_1 = \frac{A}{|B_{r_1}|} + 1$.

By Corollary 3.8 and (3.33), we have

$$\begin{aligned} \|v\|_{L^\infty(B_{r_1}(x_1))} &\leq \|v\|_{L^\infty(B_{3r_1}(x_0))} \\ &\leq Q_\infty \|v\|_{L^\infty(B_{r_1}(x_0))}^\beta \|v\|_{L^\infty(B_{4r_1}(x_0))}^{1-\beta} \\ &\leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^\beta (\varepsilon_0 + E_0), \end{aligned}$$

where C depends on Q_∞ and r_1 . By iterated application of Corollary 3.8 to v with radii r_1 , $3r_1$ and $4r_1$ over the chain of spheres, we have, by (3.32),

$$\begin{aligned} \|v\|_{L^\infty(B_{r_1}(x_k))} &\leq Q_\infty \|v\|_{L^\infty(B_{r_1}(x_{k-1}))}^\beta \|v\|_{L^\infty(B_{4r_1}(x_{k-1}))}^{1-\beta} \\ &\leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\beta^{N_1}} (\varepsilon_0 + E_0), \end{aligned}$$

where C depends on Q_∞ and r_1 . Now, we let $\tilde{x} = P_{M+1} - r\nu(P_{M+1})$ where $r < 2r_1$. Using Corollary 3.8 again for spheres centered at x_k of radii r_1 , $3r_1 - r$ and $3r_1 - \frac{r}{2}$, we obtain that

$$\begin{aligned} \|v\|_{L^\infty(B_{3r_1-r}(x_k))} &\leq Q_\infty \|v\|_{L^\infty(B_{r_1}(x_k))}^{\tau_r} \|v\|_{L^\infty(B_{3r_1-\frac{r}{2}}(x_k))}^{1-\tau_r} \\ &\leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E_0) r^{(1-\tau_r)\gamma}, \end{aligned}$$

which completes the proof. \square

REMARK 3.10. *Let us observe that, in order to apply Proposition 3.9 to the singular function defined in Section 4 when $n = 2, 4$, we need to replace the condition (3.33) by*

$$(3.35) \quad |v(x)| \leq (\varepsilon_0 + E_0) |\ln(\text{dist}(x, \Sigma_{M+1}))|^{\frac{1}{2}}, \quad x \in K.$$

By using the same proof technique, we can obtain the same result with (3.34) replaced by

$$(3.36) \quad |v(\tilde{x})| \leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E_0} \right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E_0) |\ln r|^{\frac{1-\tau_r}{2}}.$$

4. Proof of the main result. Assume that D_M is the subdomain of the partition of Ω where the maximum of $\|q^{(1)} - q^{(2)}\|$ is realized and let us denote

$$(4.1) \quad E = \|q^{(1)} - q^{(2)}\|_{L^\infty(D_M)} = \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)}.$$

We consider the chain of domains, D_0, D_1, \dots, D_M , as before; S, K and K_0 are defined as in the previous section. We set

$$U_0 = \Omega, \quad U_k = \Omega \setminus \bigcup_{j=1}^k D_j, \quad k = 1, \dots, M \quad \text{and} \quad W_k = \bigcup_{j=0}^k D_j.$$

Let $y \in K$. For dimension $n \geq 4$, let $G_1(x, y)$ and $G_2(x, y)$ be the Green's function related to $q^{(1)}$ and $q^{(2)}$, respectively, the existence and behavior of which was shown in Proposition 3.3. For dimension $n = 2, 3$, let $G_1(x, y)$ and $G_2(x, y)$ be a first order derivative of the Green's function, the existence and behavior of which was shown in Propositions 3.5 and 3.4, respectively. We define

$$(4.2) \quad S_k(y, z) = \int_{U_k} (q^{(1)} - q^{(2)})(x) G_1(x, y) G_2(x, z) dx.$$

By Proposition 3.3, 3.4 and 3.5, there exist a constant C such that

$$(4.3) \quad \begin{aligned} |S_k(y, z)| &\leq CE |\ln(\text{dist}(y, U_k)) \ln(\text{dist}(z, U_k))|^{\frac{1}{2}}, & y, z \in K \cap W_k, \quad n = 2, 4; \\ |S_k(y, z)| &\leq CE (\text{dist}(y, U_k) \text{dist}(z, U_k))^{-\frac{1}{2}}, & y, z \in K \cap W_k, \quad n = 3; \\ |S_k(y, z)| &\leq CE (\text{dist}(y, U_k) \text{dist}(z, U_k))^{2-\frac{n}{2}}, & y, z \in K \cap W_k, \quad n \geq 5. \end{aligned}$$

We focus on $n = 3$ first; we will discuss the adaptation of the proof for the case $n = 2, 4$ and $n \geq 5$ at the end of the proof.

LEMMA 4.1. *For every $y, z \in K \cap W_k$, we have $S_k(\cdot, z), S_k(y, \cdot) \in H^1(K \cap W_k)$ and*

$$(4.4) \quad (-\Delta + q^{(1)})S_k(\cdot, z) = 0, \quad (-\Delta + q^{(2)})S_k(y, \cdot) = 0 \quad \text{in } K \cap W_k.$$

The proof of this Lemma follows from the symmetry of G_i ($i = 1, 2$) and changing the order of integration and differentiation.

LEMMA 4.2. *If for some $\varepsilon_0 > 0$ and $k \in \{1, \dots, M-1\}$ we have that*

$$(4.5) \quad |S_k(y, z)| \leq \varepsilon_0, \quad \forall y, z \in K_0,$$

then

$$(4.6) \quad \begin{aligned} |S_k(y_r, y_r)| &\leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E) |\ln r|, & n = 2 \text{ or } 4, \\ |S_k(y_r, y_r)| &\leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E) r^{-1}, & n = 3, \\ |S_k(y_r, y_r)| &\leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E) r^{4-n}, & n \geq 5, \end{aligned}$$

where $y_r = P_{k+1} - r\nu(P_{k+1})$, r is small, $\nu(P_{k+1})$ is the exterior unit normal vector to ∂D_k at P_{k+1} and the positive constant C depends on r_0, L, A, B and n .

Proof. Let the dimension $n = 3$. We fix $z \in K_0$ first and consider $v(y) = S_k(y, z)$. By Lemma 4.1, v solves the equation $(-\Delta + q^{(1)})v = 0$ in $K \cap W_k$. Moreover, by (3.14), we have

$$(4.7) \quad |v(y)| \leq C E \text{dist}(y, \Sigma_{k+1})^{-\frac{1}{2}}, \quad y \in K \cap W_k.$$

Then, by Proposition 3.9 with $\gamma = -\frac{1}{2}$, we have, for $0 < r < 2r_1$,

$$(4.8) \quad |S_k(y_r, z)| \leq C \left(\frac{\varepsilon_0}{\varepsilon_0 + E} \right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E) r^{-\frac{1}{2}}.$$

Next, we consider

$$(4.9) \quad \tilde{v}(z) = S_k(y_r, z), \quad z \in K \cap W_k,$$

which solves the equation $(-\Delta + q^{(2)})\tilde{v} = 0$ in $K \cap W_k$, and, by (3.14), satisfies

$$(4.10) \quad |\tilde{v}(z)| \leq C E (r \operatorname{dist}(z, \Sigma_{k+1}))^{-\frac{1}{2}}, \quad z \in K \cap W_k.$$

By Proposition 3.9, again, we then obtain estimate (4.6) for $n = 3$.

The proof for other dimensions follows from the same proof with a few modifications. For $n = 2, 4$, a modified version of Proposition 3.9, as stated in Remark 3.10, needs to be applied. For $n \geq 5$, one can apply Proposition 3.9 with $\gamma = 2 - \frac{n}{2}$. \square

Proof of Theorem 2.7. Let

$$\varepsilon = \|\Lambda_1^{(\Sigma)} - \Lambda_2^{(\Sigma)}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})}$$

and

$$\delta_k = \|q^{(1)} - q^{(2)}\|_{L^\infty(W_k)}, \quad k = 0, 1, \dots, M.$$

From the Alessandrini identity (see for instance, Chapter 5 of [9])

$$(4.11) \quad \int_{\Omega} (q^{(1)} - q^{(2)})(x) G_1(x, y) G_2(x, z) dx = \langle (\Lambda_1 - \Lambda_2) G_1(\cdot, y), \overline{G}_2(\cdot, z) \rangle, \quad \forall y, z \in K_0$$

and Proposition 3.3, we find that

$$(4.12) \quad |S_{k-1}(y, z)| \leq C (\varepsilon + \delta_{k-1}).$$

Let $P_k \in \Sigma_k$ and $y_r = z_r = P_k - r\nu(P_k)$, where $\nu(P_k)$ is the exterior unit normal vector to ∂D_{k-1} and r is small. We write

$$(4.13) \quad S_{k-1}(y_r, y_r) = I_1 + I_2$$

with

$$(4.14) \quad I_1 = \int_{B_{\rho_0}(P_k) \cap D_k} (q^{(1)} - q^{(2)})(x) G_1(x, y_r) G_2(x, y_r) dx$$

and

$$(4.15) \quad I_2 = \int_{U_{k-1} \setminus (B_{\rho_0}(P_k) \cap D_k)} (q^{(1)} - q^{(2)})(x) G_1(x, y_r) G_2(x, y_r) dx,$$

where $\rho_0 = \frac{r_0}{6}$.

For $n = 3$, by Proposition 3.4, we have

$$(4.16) \quad |I_2| \leq C E.$$

We estimate I_1 as follows:

$$\begin{aligned}
(4.17) \quad |I_1| &= |q_k^{(1)} - q_k^{(2)}| \left| \int_{B_{\rho_0}(P_k) \cap D_k} G_1(x, y_r) G_2(x, y_r) dx \right| \\
&\geq |q_k^{(1)} - q_k^{(2)}| \left\{ \left| \int_{B_{\rho_0}(P_k) \cap D_k} \Gamma(x, y_r) \Gamma(x, y_r) dx \right| \right. \\
&\quad - \left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_1(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right| \\
&\quad - \left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_2(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right| \\
&\quad \left. - \left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_1(x, y_r) - \Gamma(x, y_r)) (G_2(x, y_r) - \Gamma(x, y_r)) dx \right| \right\}.
\end{aligned}$$

By Propositions 3.4 and the fact that

$$\begin{aligned}
&\left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_i(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right| \\
&\leq \frac{1}{2} \int_{B_{\rho_0}(P_k) \cap D_k} \left(2|G_i(x, y_r) - \Gamma(x, y_r)|^2 + |\Gamma(x, y_r)|^2 \right) dx, \quad i = 1, 2
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_1(x, y_r) - \Gamma(x, y_r)) (G_2(x, y_r) - \Gamma(x, y_r)) dx \right| \Big\} \\
&\leq \frac{1}{2} \int_{B_{\rho_0}(P_k) \cap D_k} (|G_1(x, y_r) - \Gamma(x, y_r)|^2 + |G_2(x, y_r) - \Gamma(x, y_r)|^2) dx,
\end{aligned}$$

we obtain that

$$|I_1| \geq |q_k^{(1)} - q_k^{(2)}| \left(\frac{1}{2} \int_{B_{\rho_0}(P_k) \cap D_k} |\Gamma(x, y_r)|^2 dx - C \right).$$

Using the explicit form of $\Gamma(x, y)$, we find that

$$\begin{aligned}
(4.18) \quad |I_1| &\geq |q_k^{(1)} - q_k^{(2)}| (Cr^{-1} - C) \\
&\geq C|q_k^{(1)} - q_k^{(2)}| r^{-1} - CE.
\end{aligned}$$

Now, by Lemma 4.2 and (4.12), we have

$$|S_{k-1}(y_r, y_r)| \leq C \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon + \delta_{k-1} + E) r^{-1}.$$

Hence, using (4.13), (4.16) and (4.18), we have

$$|q_k^{(1)} - q_k^{(2)}| r^{-1} \leq C \left(E + \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon + \delta_{k-1} + E) r^{-1} \right),$$

so that

$$(4.19) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left(\left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} + r \right).$$

Noting that

$$\tau_r = \frac{\ln \left(\frac{12r_1 - 2r}{12r_1 - 3r} \right)}{\ln \left(\frac{6r_1 - r}{2r_1} \right)}, \quad \forall r \in (0, 2r_1)$$

implies

$$\frac{\tau_r}{r} \geq \frac{1}{12r_1 \ln 3}, \quad \forall r \in (0, 2r_1)$$

we get

$$(4.20) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left(\left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\beta^{2N_1} (12r_1 \ln 3)^{-2} r^2} + r \right).$$

By taking $r = \left| \ln \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-1/4}$ and noting that

$$\left(e^{-r^{-4}} \right)^{\beta^{2N_1} (12r_1 \ln 3)^{-2} r^2} \leq Cr, \quad \forall r > 0$$

for some constant C , we obtain that

$$(4.21) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left| \ln \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-\frac{1}{4}}.$$

We let

$$\omega(t) = \begin{cases} |\ln t|^{-\frac{1}{4}}, & 0 < t < e^{-3}, \\ 3^{-\frac{1}{4}}, & t \geq e^{-3}. \end{cases}$$

Noting that the function $t \mapsto t\omega_n(1/t)$ is increasing, we have

$$\frac{\varepsilon + \delta_{k-1} + E}{\varepsilon + \delta_{k-1}} \omega \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \geq \omega(1),$$

hence

$$\delta_{k-1} \leq \varepsilon + \delta_{k-1} \leq (\omega(1))^{-1} (\varepsilon + \delta_{k-1} + E) \omega \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right),$$

which with (4.21) gives that

$$(4.22) \quad \delta_k \leq C(\varepsilon + \delta_{k-1} + E) \omega \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right).$$

The above choice of r is possible only if

$$\left| \ln \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-1/4} < 2r_1.$$

However, if

$$\left| \ln \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{-1/4} \geq 2r_1,$$

that is,

$$\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \geq e^{-(2r_1)^{-4}},$$

the fact that

$$\sup_{\substack{r \in (0, 2r_1) \\ t \in (e^{-(2r_1)^{-4}}, 1)}} t^{\beta^{2N_1} (12r_1 \ln 3)^{-2} r^2} |\ln t|^{\frac{1}{4}}$$

is finite shows that (4.21) still holds true, then (4.22) follows.

We iterate (4.22), starting from $\delta_0 = 0$, and find

$$(4.23) \quad \delta_k + \varepsilon \leq (C + 3^{1/4})^k (E + \varepsilon) \omega_k \left(\frac{\varepsilon}{\varepsilon + E} \right),$$

where ω_k is the composition of ω k times with itself. We recall that $E = \delta_M$, whence,

$$(4.24) \quad E + \varepsilon \leq (C + 3^{1/4})^M (E + \varepsilon) \omega_M \left(\frac{\varepsilon}{\varepsilon + E} \right),$$

so that

$$(4.25) \quad E \leq \frac{1 - \omega_M^{-1}((C + 3^{1/4})^{-M})}{\omega_M^{-1}((C + 3^{1/4})^{-M})} \varepsilon,$$

which completes the proof for dimension $n = 3$.

The proof for $n = 2$ and $n = 4$ follows from a careful inspection and adaptation of the above proof for $n = 3$. By Proposition 3.5 and 3.4, and the explicit form of

$$\begin{aligned} \Gamma(x, y) &= -\frac{2\pi(x_2 - y_2)}{|x - y|^2}, & n = 2, \\ \Gamma(x, y) &= -\frac{1}{4\pi^2|x - y|^2}, & n = 4, \end{aligned}$$

we obtain that

$$(4.26) \quad |q_k^{(1)} - q_k^{(2)}| \leq C(\varepsilon + \delta_{k-1} + E) \left(\left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right)^{\tau_r^2 \beta^{2N_1}} + |\ln r|^{-1} \right).$$

Then, by taking $r = \frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E}$ and adapting the function $\omega(t)$ according to

$$\tilde{\omega}(t) = \begin{cases} |\ln t|^{-1}, & 0 < t < e^{-2}, \\ \frac{1}{2}, & t \geq e^{-2}, \end{cases}$$

we end up with

$$(4.27) \quad E \leq \frac{1 - \tilde{\omega}_M^{-1}((C + 2)^{-M})}{\tilde{\omega}_M^{-1}((C + 2)^{-M})} \varepsilon,$$

which completes the proof for $n = 2$ and $n = 4$.

Let us sketch the required modifications of the proof for higher dimensional cases ($n \geq 5$) below. First, one can use the same decomposition of the singular function as in (4.13), (4.14) and (4.15), and by Proposition 3.3, the same upper bound estimate of I_2 as in (4.16) is obtained. Then, because the order of the singularity of $\Gamma(\cdot, y)$ increases as the dimension n increases, $G(\cdot, y_r) - \Gamma(\cdot, y_r)$ may not be uniformly bounded in $L^2(B_{\rho_0}(P_k) \cap D_k)$ with respect to r . A feasible modification here is to compare the orders of singularity of $G(\cdot, y_r) - \Gamma(\cdot, y_r)$ and $\Gamma(\cdot, y_r)$. More precisely, we can estimate

$$\left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_i(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right|, \quad i = 1, 2,$$

using Hölder inequality, as

$$\begin{aligned} & \left| \int_{B_{\rho_0}(P_k) \cap D_k} (G_i(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r) dx \right| \\ & \leq \int_{B_{\rho_0}(P_k) \cap D_k} |(G_i(x, y_r) - \Gamma(x, y_r)) \Gamma(x, y_r)| dx \\ & \leq \|G_i(\cdot, y_r) - \Gamma(\cdot, y_r)\|_{L^2(\Omega_0)} \|\Gamma(\cdot, y_r)\|_{L^2(B_{\rho_0}(P_k) \cap D_k)}. \end{aligned}$$

Substituting the above inequality into (4.17) and noting the positiveness of $\Gamma(\cdot, y_r)$, we obtain the estimate of the lower bound of I_1 as

$$\begin{aligned} (4.28) \quad |I_1| & \geq |q_k^{(1)} - q_k^{(2)}| (\|\Gamma(\cdot, y_r)\|_{L^2(B_{\rho_0}(P_k) \cap D_k)}^2 \\ & \quad - \|G_1(\cdot, y_r) - \Gamma(\cdot, y_r)\|_{L^2(\Omega_0)} \|\Gamma(\cdot, y_r)\|_{L^2(B_{\rho_0}(P_k) \cap D_k)} \\ & \quad - \|G_2(\cdot, y_r) - \Gamma(\cdot, y_r)\|_{L^2(\Omega_0)} \|\Gamma(\cdot, y_r)\|_{L^2(B_{\rho_0}(P_k) \cap D_k)} \\ & \quad - \|G_1(\cdot, y_r) - \Gamma(\cdot, y_r)\|_{L^2(\Omega_0)} \|G_2(\cdot, y_r) - \Gamma(\cdot, y_r)\|_{L^2(\Omega_0)} - C). \end{aligned}$$

By the explicit form of $\Gamma(x, y)$ and Proposition 3.3, especially (3.7), we observe that $G_i(\cdot, y_r) - \Gamma(\cdot, y_r)$ has the lower order of singularity than $\Gamma(\cdot, y_r)$. Hence, by Young's inequality as in the previous proof for $n = 3$, we conclude that

$$|I_1| \geq C |q_k^{(1)} - q_k^{(2)}| r^{4-n} - CE$$

for r small. Then, following the same argument with the same value of r and noting that

$$\left(e^{-r^{-4}} \right)^{\beta^{2N_1} (12r_1 \ln 3)^{-2} r^2} \leq C r^{n-4}, \quad \forall r > 0$$

holds true for any $n \geq 5$, we obtain that

$$(4.29) \quad |q_k^{(1)} - q_k^{(2)}| \leq C (\varepsilon + \delta_{k-1} + E) \left| \ln \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right|^{\frac{4-n}{4}}.$$

The last step of the modifications is to adapt the function $\omega(t)$ according to

$$\tilde{\omega}(t) = \begin{cases} |\ln t|^{\frac{4-n}{4}}, & 0 < t < e^{-n}, \\ n^{\frac{4-n}{4}}, & t \geq e^{-n}. \end{cases}$$

Then we end up with

$$(4.30) \quad E \leq \frac{1 - \tilde{\omega}_M^{-1}((C + n^{\frac{n-4}{4}})^{-M})}{\tilde{\omega}_M^{-1}((C + n^{\frac{n-4}{4}})^{-M})} \varepsilon,$$

which completes the proof for $n \geq 5$. \square

5. Exponential behavior of the Lipschitz stability constant. In this section, we give a model example to show that the Lipschitz stability constant $C = C(n, r_0, L, A, N)$ in Theorem 2.7 behaves exponentially with respect to the number N of the subdomains. The construction is an analogue of the construction in [11], pertaining to the inverse conductivity problem.

Let Ω be the unit ball $B_1(0) \subset \mathbb{R}^n$ and $D = [-1/2, 1/2]^n$ be the cube of side 1 centered at the origin. We define the class of admissible potentials by

$$(5.1) \quad \mathcal{A} = \{q \in L^\infty(\Omega) \mid 1/2 \leq q \leq 3/2 \text{ in } \Omega \text{ and } q = 1 \text{ in } \Omega \setminus D\}$$

and denote the operator from potential q to Λ_q by F , which maps \mathcal{A} into $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. We fix a positive integer N and let N_1 be the smallest integer such that $N \leq N_1^n$. We divide each side of the cube D into N_1 equal parts of length $h = 1/N_1$ and let S_{N_1} be the set of all open cubes of the type

$$D' = (-1/2 + (j'_1 - 1)h, -1/2 + j'_1 h) \times \cdots \times (-1/2 + (j'_n - 1)h, -1/2 + j'_n h),$$

where j'_1, \dots, j'_n are integers belonging to $\{1, \dots, N_1\}$. We order such cubes as follows. For any two different cubes D' and D'' belonging to S_{N_1} , we say that $D' \prec D''$ if and only if there exists an $i_0 \in \{1, \dots, n\}$ such that $j'_i = j''_i$ for any $i < i_0$ and $j'_{i_0} < j''_{i_0}$. We define

$$\mathcal{A}_N = \{q \in L^\infty(\Omega) \mid q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x) + \chi_{D_0}(x), \quad q_j \in [1/2, 3/2]\}.$$

Our aim is to estimate from below the Lipschitz constant $C(N)$ in terms of N . A simple computation shows polynomial behavior of the lower bound estimate of $C(N)$. To obtain the exponential estimate, we then need to employ a topological argument.

Consider a subset $\tilde{\mathcal{A}}_N \subset \mathcal{A}_N$ defined by

$$\tilde{\mathcal{A}}_N = \{q \in L^\infty(\Omega) \mid q(x) = \sum_{j=1}^N q_j \chi_{D_j}(x) + \chi_{D_0}(x), \quad q_j \in \left\{\frac{1}{2}, 1, \frac{3}{2}\right\}\}.$$

It is easy to check that $\tilde{\mathcal{A}}_N$ is a $1/2$ -net of \mathcal{A}_N with 3^N elements and, for any two different $q_1, q_2 \in \tilde{\mathcal{A}}_N$, we have $\|q_1 - q_2\|_{L^\infty(\Omega)} = 1/2$. Based on Mandache's result [10, Lemma 3], there exist a constant K , which only depends on dimension n , such that for every $\varepsilon \in (0, e^{-1})$, there is an ε -net Y for $F(\mathcal{A})$ with at most $e^{K(-\ln \varepsilon)^{2n-1}}$ elements. For $\varepsilon \in (0, e^{-1})$ and $N \in \mathbb{N}$ let

$$Q(\varepsilon, N) = e^{K(-\ln \varepsilon)^{2n-1}}.$$

Note that

$$3^N > e^{K(-\ln \varepsilon)^{2n-1}}$$

if

$$\varepsilon > e^{-K_1 N^{1/(2n-1)}} = \varepsilon_0(N)$$

where $K_1 = (K^{-1} \ln 3)^{1/(2n-1)}$. There exists N_0 such that for $N \geq N_0$ we have that $\varepsilon < e^{-1}$. Thus, for $N \geq N_0$, if we take $\varepsilon = \varepsilon_0$ we have $3^N > Q(\varepsilon, N)$. Then, there exist two different $q_1, q_2 \in \tilde{\mathcal{A}}_N$ such that $\|q_1 - q_2\|_{L^\infty(\Omega)} = 1/2$ with their images under F in the same ball of radius ε centered at a point of Y , that is,

$$\frac{1}{2} = \|q_1 - q_2\|_{L^\infty(\Omega)} \leq C_N \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))} \leq 2C_N \varepsilon_0(N)$$

from which we get

$$C(N) \geq \frac{1}{4} e^{K_1 N^{1/(2n-1)}}.$$

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